IRREDUCIBILITY OF SOME QUANTUM REPRESENTATIONS OF MAPPING CLASS GROUPS

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Abstract. The $SU(2)$ TQFT representation of the mapping class group of a closed surface of genus g, at a root of unity of prime order, is shown to be irreducible. Some examples of reducible representations are also given.

1. Introduction

The Witten-Reshetikhin-Turaev topological quantum field theories (see Reshetikhin and Turaev [RT] or Turaev's book [T]) provide many interesting finite-dimensional representations of mapping class groups of surfaces, about which little is currently known. In this paper we will consider only the representations coming from the $SU(2)$ theory, which may be defined and studied via the Kauffman bracket skein theory (see Lickorish [L], Roberts [R1]). Calculations using this approach are easier and more concrete than in the more general cases (where one has to work more explicitly with quantum groups) but typically provide insight into the general cases, which can be worked out along the same lines (as for example with the integrality results of Masbaum-Roberts [MR] and then Masbaum-Wenzl [MW]).

Recent papers by Funar [Fun] and Masbaum [M] studied the question of whether the image of the mapping class group of a closed surface under such a representation (at an rth root of unity) was infinite or not. Here, another aspect will be considered: are the representations irreducible? I was asked this question by Ivan Smith, who was interested in the geometric quantization approach to the representations, and found it difficult to answer using such an algebro-geometric approach. Surprisingly, even the TQFT literature does not seem to provide the answer. The purpose of this note, therefore, is to make a start by answering the question at least in some cases.

Theorem. *Let $r \geq 3$ be prime. Then the $SU(2)$ TQFT representation of the mapping class group of a closed surface of genus g, at an rth root of unity, is irreducible.*

Unfortunately the proof which will be explained below does not seem to generalise in a straightforward way (unlike, say, the methods of [R2]) to either non-prime r or to higher rank quantum groups. Neither is it completely clear how to extend it to the case of surfaces with punctures.

2. Skein theory preliminaries

The proof of the theorem is not very long or complicated, so this explanation of the background will be kept equally brief. The main purpose is simply to fix the notation. The

Date: September 21, 1999.

paper by Lickorish [L] and the book of Kauffman and Lins [KL] contain full explanations of how one uses skein theory to build 3-manifold invariants, whilst [BHMV, R3] explain how to develop a full TQFT using the same principles.

Fix $r \geq 3$, the integer 'level', and let $A = e^{2\pi i/4r}$. The symbol $\mathcal{S}M$ denotes the Kauffman skein space (a complex vector space defined using the parameter A) of a compact oriented 3-manifold M. It is the vector space generated by isotopy classes (rel boundary) of framed links inside M, modulo the usual local Kauffman bracket relations.

The Jones-Wenzl idempotents $f^{(a)}$ of the Temperley-Lieb algebras (skein spaces of a cylinder with a points at each end) are defined for $a = 0, 1, \dots, r-1$. Inside any skein space $\mathcal{S}M$ one can consider the subspace spanned by elements consisting of $f^{(r-1)}$ in a small cylinder with the ends connected up in any way. Factoring out by this subspace gives the *reduced skein space* $\mathcal{R}M$. It is a fact (see Roberts [R3]) that the reduced skein space of any 3-manifold depends only on its boundary, and gives a model for the Witten-Reshetikhin-Turaev vector space of this boundary.

In particular if H is a handlebody and Σ its boundary then the reduced skein space $\mathcal{R}H$ is identified with the W-R-T space usually written (for example in Blanchet, Habegger, Masbaum and Vogel [BHMV]) as $V(\Sigma)$. In [BHMV], the space $V(\Sigma)$ is constructed as a quotient of $\mathcal{S}H$, but in a fairly abstract way; the point of the reduced skein space is simply that it is an explicit local combinatorial description of the quotient.

The most important such spaces are those associated to a solid torus, denoted $\mathcal{S}T$ and $\mathcal{R}T$ for convenience. The skein space $\mathcal{S}T$ is a polynomial algebra with one generator α, a single curve winding once around the torus. The elements $\phi_a \in \mathcal{S}T$ ($a = 0, 1, \dots, r-2$), given by taking the ath Chebyshev polynomial of α (or by closing up Jones-Wenzl idempotents) are particularly important, as they descend to a basis for the $(r-1)$-dimensional space $\mathcal{R}T$. Lickorish's construction of 3-manifold invariants is based on an element $\Omega \in \mathcal{R}T$ defined by

$$\Omega = \eta \sum_{a=0}^{r-2} \Delta(a)\phi_a,$$

where

$$\eta = \frac{A^2 - A^{-2}}{i\sqrt{2r}} = \sqrt{2/r}\sin(\pi/r) \quad \text{and } \Delta(a) = (-1)^a \frac{A^{2(a+1)} - A^{-2(a+1)}}{A^2 - A^{-2}}.$$

(In [BHMV] and [MR], the symbol ω was used instead of Ω; the convention here agrees with the one in [R2].)

For a handlebody H of genus $g \geq 2$, one can again write down a basis of $\mathcal{R}H$. The usual basis is given by picking $3g-3$ discs chopping up H in a pants decomposition, and then drawing a trivalent spine dual to the decomposing discs. The standard basis elements v are made by attaching Jones-Wenzl idempotents to the edges of this graph and joining them suitably at the vertices. They are parametrised by the labellings of their idempotents, in other words by a subset of the set of labellings of the edges by integers in the range 0 to $r-2$. The vacuum vector v_0 is the basis vector corresponding to the empty link (all labels are 0).

The action of the mapping class group Γ_g on $\mathcal{R}H$ can be defined in a natural but implicit way (see [R1]), but here it is more useful to have an explicit description of the action of

a positive Dehn twist T_γ about a curve $\gamma \in \Sigma$. If x is an element of $\mathcal{R}H$ then $T_\gamma x$ is represented by adjoining to a skein element representing x the curve γ with Ω inserted onto it (with framing -1 relative to the surface), drawn just inside the boundary of H.

In particular, if γ is the boundary of one of the pants discs, the twist T_γ acts on a standard basis vector v by putting a full positive twist in the edge passing through this disc. Idempotents are eigenvectors under such twist operations. Consequently, if the relevant edge is coloured with a, then $T_\gamma v = \xi_a v$, where $\xi_a = (-1)^a A^{a^2+2a}$ is the associated eigenvalue (twist coefficient).

Remark. The representation defined by these twist generators (or as constructed in [R1]) is only projective, and it is customary to lift to a genuine linear representation of a central extension of Γ_g. However, the projective ambiguity has no bearing on the question of irreducibility, so this fact can be safely ignored.

3. Proof of the theorem

Lemma 1. *If r is prime then the vectors t_b, for $b = 0, 1, \ldots, r-2$, given by placing b parallel -1-framed copies of Ω in the solid torus form a basis for $\mathcal{R}T$.*

Proof. The quickest way to see this is to use the non-degenerate pairing $\mathcal{R}T \times \mathcal{R}T \to \mathbb{C}$ obtained by gluing together two solid tori to make S^3 (whose skein space is canonically $\mathbb{C}$). Pairing t_b with ϕ_a results in $\xi_b^a \Delta(a)$, so that the change of basis matrix expressing the t_b, viewed as linear functionals, in terms of the dual functionals ϕ_a^* is a Vandermonde matrix (ξ_a^b) times a diagonal matrix whose diagonal entries are the (non-zero) $\Delta(a)$. Now $\xi_a = (-1)^a A^{a^2+2a}$, and one can easily check that these are all distinct when r is prime, hence the first matrix is invertible. The second is obviously invertible, and so the t_b indeed form a basis. □

Lemma 2. *Suppose C is a collection of disjoint curves on Σ_g. Then one can obtain an element of $\mathcal{R}H_g$ by viewing these curves as lying in H_g and attaching Ω to each one with framing -1 relative to the surface. Such vectors $\Omega(C)$ span $\mathcal{R}H_g$. Therefore the image of the group algebra of Γ_g applied to the vacuum vector v_0 is all of $\mathcal{R}H_g$.*

Proof. Consider H_g as a thickened $(g-1)$-holed disc. The reduced skein space is spanned by a finite number of elements of $\mathcal{S}H_g$, which may be represented by links lying in the holed disc (consider the usual planar projection onto this disc). Given such a planar link L, isotop it to be near to the boundary Σ_g, and rewrite each of its components (thought of as an element of $\mathcal{R}T$) as a linear combination of the basis elements t_b. Since each t_b is really $\Omega(C)$, for C a single curve parallelled b times, this proves the assertion about spanning. The final part follows immediately from the description of the action of a Dehn twist on $\mathcal{R}H_g$. □

Note that this does not immediately imply irreducibility. For example, if $\mathbb{Z}$ acts on $\mathbb{C}^2$ with distinct eigenvalues then the action is reducible but the group algebra applied to any non-eigenvector is all of $\mathbb{C}^2$. (This example can even be chosen to be unitary.)

Lemma 3. *The subgroup P_g of the mapping class group generated by Dehn twists on the standard pants curves is a free abelian group, under whose action $\mathcal{R}H_g$ breaks up as a sum of one-dimensional representations, spanned by the standard basis vectors v.*

Proof. The standard basis vectors are certainly simultaneous eigenvectors for the twists generating P_g, as each twist just multiplies the vector by a twist coefficient ξ_a. But their collections of eigenvalues are distinct since the ξ_a are (when r is prime) and hence they span individual one-dimensional eigenspaces. □

Now the theorem can be proved. Suppose θ is any endomorphism of $\mathcal{R}H_g$ commuting with the action of the whole mapping class group Γ_g. Then it certainly acts diagonally with respect to the standard basis, because it commutes in particular with the subgroup P_g whose eigenvectors they are. Let us write $\theta v = \lambda_v v$ for a standard basis vector v. All we need to do is to show that the matrix of θ is actually a *scalar* to conclude, via Schur's lemma, that $\mathcal{R}H_g$ is an irreducible representation of Γ_g. To see this, observe that any standard basis vector v can be generated from the vacuum vector v_0 by the action ψ of some element of the group algebra of Γ_g, by lemma 2. Then, since θ commutes with ψ,

$$\theta v = \theta\psi v_0 = \psi\theta v_0 = \lambda_{v_0}\psi v_0 = \lambda_{v_0} v,$$

but also $\theta v = \lambda_v v$, so θ is a scalar.

4. Further comments

There are certainly cases in which the representations are not irreducible. It is difficult to find these in general, but in genus 1, there is a large body of literature studying modular invariant partition functions for affine Lie algebras which provides a more than complete solution. See Capelli-Itzykson-Zuber [CIZ], Fuchs' book [Fuc], and Gannon [Ga] for a short proof of the result of [CIZ].

The problem studied in these references is to find all $SL(2,\mathbb{Z})$-invariant linear combinations

$$Z = \sum_{a,b=0}^{r-2} Z_{a,b}\phi_a \otimes \bar{\phi}_b \in V \otimes \bar{V},$$

where $V = V(\Sigma_1)$ (and $SL(2,\mathbb{Z}) = \Gamma_1$ is the mapping class group of the torus acting on it), and the coefficients $Z_{a,b}$ are non-negative integers. Since $\bar{V} \cong V^*$, such elements can be thought of as invariant endomorphisms of V, and a reasonable first step in classifying such elements Z is to find the commutant of $SL(2,\mathbb{Z})$ in $\mathrm{End}(V)$. This is carried out by Cappelli, Itzykson and Zuber [CIZ], who find the dimension of the commutant in terms of the number of divisors of r. (They then go on to find the non-negative integer matrices Z lying in the commutant and to obtain an amazing $A-D-E$ classification.) In particular, the commutant is trivial when r is prime, which agrees with the irreducibility theorem proved above, and also shows that it is sharp in genus 1.

The higher-genus situation does not seem to have been studied much. It is worth observing that the method of [CIZ] for finding the commutant in genus 1 relies on averaging over the image of $\Gamma_1 = SL(2,Z)$, which is finite (see for example Gilmer [Gi]). In higher genus, as noted in the introduction, the image is known to be (with finitely many small r exceptions) infinite, so such methods fail. Whether one can apply the methods of skein theory to the problem of modular invariants, or vice versa, remains to be seen.

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